

THE DISTRIBUTION OF ZEROS OF $\zeta'(s)$ AND GAPS BETWEEN ZEROS OF $\zeta(s)$

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ABSTRACT. Assume the Riemann Hypothesis, and let $\gamma^+ > \gamma > 0$ be ordinates of two consecutive zeros of $\zeta(s)$. It is shown that if $\gamma^+ - \gamma < v/\log \gamma$ with $v < c$ for some absolute positive constant c , then the box

$$\{s = \sigma + it : 1/2 < \sigma < 1/2 + v^2/4 \log \gamma, \gamma \leq t \leq \gamma^+\}$$

contains exactly one zero of $\zeta'(s)$. In particular, this allows us to prove half of a conjecture of Radziwiłł [14] in a stronger form. Some related results on zeros of $\zeta(s)$ and $\zeta'(s)$ are also obtained.

1. INTRODUCTION

Throughout this paper $s = \sigma + it$ is a complex variable. Write $\rho = \beta + i\gamma$ and $\rho' = \beta' + i\gamma'$ to be a generic zero of $\zeta(s)$ and $\zeta'(s)$, respectively. If $\zeta(1/2 + i\gamma) = 0$, let γ^+ denote the smallest $t > \gamma$ with $\zeta(1/2 + it) = 0$. The phrase " γ is large" stands for " γ is larger than an absolute constant". Finally, we order the ordinates of zeros of $\zeta(s)$ as $0 < \gamma_1 \leq \gamma_2 \leq \dots$, and similarly for zeros of $\zeta'(s)$.

The distribution of zeros of $\zeta'(s)$, and its relationship to zeros of $\zeta(s)$, have been investigated by many authors (see [1], [2], [5], [6], [8], [9], [10], [11], [12], [14], [15], [16], [19]). For example, a well-known theorem of A. Speiser [15] states that the Riemann Hypothesis (RH) is equivalent to $\zeta'(s)$ having no zeros in $0 < \sigma < 1/2$.

In [16] K. Soundararajan raised the following conjecture.

Conjecture. (*Soundararajan.*) Assume RH. The following two statements are equivalent:

- (A) $\liminf_{\gamma' \rightarrow \infty} (\beta' - 1/2) \log \gamma' = 0;$
- (B) $\liminf_{\gamma \rightarrow \infty} (\gamma^+ - \gamma) \log \gamma = 0.$

As stated by Soundararajan, both of these two assertions are almost certainly true, and the point of this conjecture is that there might be a simple way of relating (A) and (B) without reference to their individual validity.

In [19] Y. Zhang showed that on RH (B) implies (A), which solved one direction of Soundararajan's Conjecture. By contrast, the attempts to prove that (A) implies (B) have been unsuccessful (see Theorem 3 below).

As an alternative, D. W. Farmer and H. Ki [5] considered the following. Let $w(x)$ be the indicator function of the unit interval $[0, 1]$, and define

$$m'(v) = \liminf_{T \rightarrow \infty} \frac{2\pi}{T \log T} \sum_{T \leq \gamma' \leq 2T} w\left(\frac{(\beta' - 1/2) \log T}{v}\right)$$

and

$$m(v) = \liminf_{T \rightarrow \infty} \frac{2\pi}{T \log T} \sum_{T \leq \gamma_n \leq 2T} w\left(\frac{(\gamma_{n+1} - \gamma_n) \log T}{2\pi v}\right).$$

They conjectured that if $m'(v) \gg v^\alpha$ for some $\alpha < 2$, then $m(v) > 0$ for all $v > 0$. This conjecture was viewed as a refinement of Soundararajan's conjecture, and was recently proved (on RH) by M. Radziwiłł [14].

On the other hand, by investigating random matrix models for $\zeta'(s)$, F. Mezzadri [13] conjectured an asymptotic formula for $m'(v)$. Namely,

$$m'(v) \sim \frac{8}{9\pi} v^{3/2}. \quad (1)$$

We also refer the readers to [4] for a detailed study in this direction. The corresponding conjecture for $m(v)$ (see e.g. [5]) is

$$m(v) \sim \frac{\pi}{6} v^3. \quad (2)$$

The validity of (1) and (2) will have significant consequences. In particular, from the work of J. B. Conrey and H. Iwaniec [3] and further work of Farmer and Ki [5], either of (1) or (2) (or even weaker forms) will imply the non-existence of Landau-Siegel zeros.

Our first result supports the relative size between $m(v)$ and $m'(v)$ in the above conjectures. In particular, we prove one direction of Radziwiłł's conjecture, which asserts that if $m(v)$ or $m'(v)$ is $\gg v^A$ then $m(v/2\pi) \asymp m'(v^2)$ (see [14]). (Note that we do not need the assumption of the lower bound for $m(v)$ or $m'(v)$.)

Theorem 1. *Assume RH. There exists an absolute constant $c > 0$, such that*

$$m(v/2\pi) \leq m'(v^2)$$

for all $v < c$.

Theorem 1 is a consequence of the following result.

Theorem 2. *Assume RH. There exists an absolute constant $c > 0$ such that for any $v < c$ the following holds: For all large γ with $\gamma^+ - \gamma < v/\log \gamma$, the box*

$$\{s = \sigma + it : \frac{1}{2} < \sigma < \frac{1}{2} + \frac{v^2}{4 \log \gamma}, \gamma \leq t \leq \gamma^+\}$$

contains exactly one zero of $\zeta'(s)$. Moreover, the zero is not on the boundary of the box.

Theorem 2 can be viewed as a refinement of a result of Zhang (see Theorem 3 in [19]). Roughly speaking, Zhang's result states that (on RH) if there occurs a small gap of consecutive zeros of $\zeta(s)$, then we may find a zero of $\zeta'(s)$ nearby. The main difference between Zhang's result and our Theorem 2 is that, Theorem 2 tells a more accurate location of ρ' : it allows us to put ρ' between consecutive zeros of $\zeta(s)$. Consequently, different pairs of (γ, γ^+) will give different ρ' 's, and this is how we shall deduce Theorem 1.

Theorem 2 should also be compared with a result of Soundararajan (see Lemma 10 in Section 2). Roughly speaking, Soundararajan's result states that there is *at most* one ρ' in a certain rectangular region between consecutive zeros of $\zeta(s)$. We will prove a complementary result (see Lemma 11 in Section 2), which asserts that there is *at least* one ρ' in a certain rectangular region. Apart from technical details, Theorem 2 is essentially a combination of these two results.

Our next result considers the unsolved part of Soundararajan's Conjecture.

Theorem 3. *Assume RH. For $\beta' > 1/2$ and $\gamma \leq \gamma' < \gamma^+$, we have*

$$\gamma^+ - \gamma \ll \sqrt{(\beta' - 1/2) \log \gamma'}.$$

In particular, we have

$$\liminf_{\substack{\gamma' \rightarrow \infty \\ \beta' > 1/2}} (\beta' - 1/2)(\log \gamma')^3 = 0 \implies \liminf_{\gamma \rightarrow \infty} (\gamma^+ - \gamma) \log \gamma = 0. \quad (3)$$

This should be compared with a result of M. Z. Garaev and C. Y. Yıldırım [10], which states that (on RH)

$$\liminf_{\gamma' \rightarrow \infty} (\beta' - 1/2)(\log \gamma')(\log \log \gamma')^2 = 0 \implies \liminf_{\gamma \rightarrow \infty} (\gamma_{n+1} - \gamma_n) \log \gamma_n = 0. \quad (4)$$

Note that the right-hand side of (4) would be trivially true if $\zeta(s)$ has infinitely many multiple zeros. Our result (3) overcomes this possible issue since $\gamma^+ > \gamma$ are ordinates of distinct zeros, but at the cost that we require a stronger assumption in the left-hand side. Note also that both (3) and (4) are weaker than Soundararajan's Conjecture.

The paper is organized as follows. In Section 2 we state preliminary results required in proving our main theorems. We will then deduce Theorem 2, Theorem 1 and Theorem 3 in order in Section 3, using results from Section 2. The last two sections are devoted to the proofs of preliminary results.

2. PRELIMINARY RESULTS

Let $\eta(s) = \pi^{-s/2} \Gamma(s/2) \zeta'(s)$, and define

$$F(t) = \begin{cases} -\Re \frac{\eta'}{\eta}(1/2 + it), & \text{if } \eta(1/2 + it) \neq 0, \\ \lim_{v \rightarrow t} F(v), & \text{otherwise.} \end{cases} \quad (5)$$

It is well-defined (see [19]), and serves as a key role in our proofs. The following five results were obtained by Zhang [19].

(i) The limit in (5) exists. Namely, $F(t)$ is well-defined. Moreover, $F(t)$ is continuous.

(ii) We have $F(t) = F_1(t) - F_2(t) + O(1)$, where

$$F_1(t) = - \sum_{\beta' > 1/2} \Re \frac{1}{1/2 + it - \rho'}, \quad (6)$$

and

$$F_2(t) = \sum_{0 < \beta' < 1/2} \Re \frac{1}{1/2 + it - \rho'}. \quad (7)$$

(iii) If $\rho = 1/2 + i\gamma$ is a simple zero of $\zeta(s)$ where $\gamma > 0$, then $F(\gamma) = \frac{1}{2} \log \gamma + O(1)$.

(iv) We have

$$\int_{\gamma}^{\gamma^+} F(t) dt \leq \pi. \quad (8)$$

(v) If both $1/2 + i\gamma$ and $1/2 + i\gamma^+$ are simple zeros of $\zeta(s)$, then

$$\int_{\gamma}^{\gamma^+} F(t) dt \equiv 0 \pmod{\pi}.$$

We require some further results of $F(t)$ given by the following four lemmas.

Lemma 4. *Let $\rho = 1/2 + i\gamma$ be a zero of $\zeta(s)$ with multiplicity $m(\rho) = m$. Then we have*

$$F(\gamma) = \frac{1}{2m} \log \gamma + O\left(\frac{1}{m}\right).$$

Lemma 5. *Let $\rho = 1/2 + i\gamma$ be a zero of $\zeta(s)$. Then we have*

$$\lim_{\substack{v \rightarrow \gamma \\ v > \gamma}} \arg \eta(1/2 + iv) \equiv \lim_{\substack{v \rightarrow \gamma \\ v < \gamma}} \arg \eta(1/2 + iv) \equiv \pi/2 \pmod{\pi}.$$

Lemma 6. *Assume RH. We have*

$$\int_{\gamma}^{\gamma^+} F(t) dt = \pi$$

for all large γ .

Lemma 7. *Assume RH, and let $F_1(t)$ be defined in (6). Then we have*

$$F(t) = F_1(t) + \log 2/2 + O(1/t).$$

For $\Re(\rho') > 1/2$, let $\theta(\rho', t_1, t_2) \in (0, \pi)$ be the argument of the angle at ρ' with rays through $1/2 + it_1$ and $1/2 + it_2$ respectively. The following lemma is also needed.

Lemma 8. *Assume RH. We have*

$$\sum_{1/2 < \Re \rho'} \theta(\rho', \gamma, \gamma^+) + \frac{\log 2}{2} (\gamma^+ - \gamma) + O\left(\frac{\gamma^+ - \gamma}{\gamma}\right) = \pi$$

for all large γ .

Next we state two useful results of Soundararajan.

Lemma 9. *For $\beta' > 1/2$ and $\gamma' > 0$, we have*

$$|\rho - \rho'| \geq \sqrt{2(\beta' - 1/2)/\log \gamma}.$$

See Lemma 2.1 in [16].

Lemma 10. *Assume RH. The box*

$$\{s = \sigma + it : 1/2 < \sigma < 1/2 + 1/\log \gamma, \gamma \leq t \leq \gamma^+\}$$

contains at most one zero of $\zeta'(s)$.

See Proposition 1.6 in [16].

The following result is also required, which can be viewed as a partial complement to Lemma 10.

Lemma 11. *Assume RH. Let a be any constant less than $\pi/3$. There exists a constant $\gamma_0(a)$ such that for $\gamma > \gamma_0(a)$ with $\gamma^+ - \gamma < a/\log \gamma$, the box*

$$\{s = \sigma + it : 1/2 < \sigma < 1/2 + 2.5(\gamma^+ - \gamma), \gamma < t < \gamma^+\}$$

contains a zero of $\zeta'(s)$.

Combining the above two results we immediately obtain

Corollary 12. *Assume RH. For large γ with $\gamma^+ - \gamma < 0.4/\log \gamma$, the box*

$$\{s = \sigma + it : 1/2 < \sigma < 1/2 + 1/\log \gamma, \gamma \leq t \leq \gamma^+\}$$

contains exactly one zero of $\zeta'(s)$. Moreover, the zero is not on the boundary of the box.

Lastly, it is interesting to record our final proposition, whose proof we shall omit because of its similarity to that of Lemma 11. (Also, we do not need it in proving our main theorems.)

Proposition 13. *Assume RH. Let Δ be any constant less than π . There exists a constant $\gamma_0(\Delta)$ such that for $\gamma > \gamma_0(\Delta)$ with $\gamma^+ - \gamma < \Delta/\log \gamma$, the strip $\{s = \sigma + it : \gamma < t < \gamma^+\}$ contains a zero of $\zeta'(s)$.*

3. PROOFS OF THEOREMS 2, 1 AND 3

Proof of Theorem 2. Let $v < 0.4$ and suppose $\gamma^+ - \gamma < v/\log \gamma$. By Corollary 12, the box

$$\{s = \sigma + it : 1/2 < \sigma < 1/2 + 1/\log \gamma, \gamma \leq t \leq \gamma^+\}$$

contains exactly one zero of $\zeta'(s)$, say $\rho' = \beta' + i\gamma'$, and it is not on the boundary. Thus, to prove the theorem, it suffices to prove that

$$\beta' - 1/2 < \frac{v^2}{4 \log \gamma}. \quad (9)$$

Note that $\beta' - 1/2 < 1/\log \gamma$. Without loss of generality, we may assume $\gamma^+ - \gamma' \geq \gamma' - \gamma$, and hence $\gamma^+ - \gamma \geq 2(\gamma' - \gamma)$.

By Lemma 9, we have

$$|\rho' - \rho| \geq \sqrt{\frac{2(\beta' - 1/2)}{\log \gamma}},$$

and this gives

$$(\gamma' - \gamma)^2 \geq \frac{2(\beta' - 1/2)}{\log \gamma} - (\beta' - 1/2)^2 \geq \frac{\beta' - 1/2}{\log \gamma},$$

since $\beta' - 1/2 < 1/\log \gamma$.

Hence, we get

$$\frac{v^2}{\log^2 \gamma} > (\gamma^+ - \gamma)^2 \geq 4(\gamma' - \gamma)^2 \geq \frac{4(\beta' - 1/2)}{\log \gamma}.$$

This gives (9) and completes the proof. □

Proof of Theorem 1. Take c to be the same as in Theorem 2, and let $v < c$. Define

$$\mathcal{S} = \{n : T \leq \gamma_n \leq 2T, \gamma_{n+1} - \gamma_n \leq \frac{v}{\log T}\}$$

and

$$\mathcal{T} = \{m : T \leq \gamma'_m \leq 2T, \beta'_m - \frac{1}{2} \leq \frac{v^2}{\log T}\}.$$

Recall that

$$m'(v) = \liminf_{T \rightarrow \infty} \frac{2\pi}{T \log T} \sum_{T \leq \gamma'_n \leq 2T} w\left(\frac{(\beta' - 1/2) \log T}{v}\right)$$

and

$$m(v) = \liminf_{T \rightarrow \infty} \frac{2\pi}{T \log T} \sum_{T \leq \gamma_n \leq 2T} w\left(\frac{(\gamma_{n+1} - \gamma_n) \log T}{2\pi v}\right),$$

where $w(x)$ is the indicator function of $[0, 1]$.

Thus, to prove $m(v/2\pi) \leq m'(v^2)$, it suffices to show that $|\mathcal{S}| - 1 \leq |\mathcal{T}|$ for all large T .

Write $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, where

$$\mathcal{S}_1 = \{n : T \leq \gamma_n \leq 2T, \gamma_{n+1} - \gamma_n = 0\},$$

and

$$\mathcal{S}_2 = \{n : T \leq \gamma_n \leq 2T, 0 < \gamma_{n+1} - \gamma_n \leq \frac{v}{\log T}\}.$$

Similarly, write $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$, where

$$\mathcal{T}_1 = \{k : T \leq \gamma'_k \leq 2T, \beta'_k - \frac{1}{2} = 0\},$$

and

$$\mathcal{T}_2 = \{k : T \leq \gamma'_k \leq 2T, 0 < \beta'_k - \frac{1}{2} \leq \frac{v^2}{\log T}\}.$$

We clearly have $|\mathcal{S}| = |\mathcal{S}_1| + |\mathcal{S}_2|$ and $|\mathcal{T}| = |\mathcal{T}_1| + |\mathcal{T}_2|$. Moreover, it is easy to see that there is a bijection between \mathcal{S}_1 and \mathcal{T}_1 . Namely, we have $|\mathcal{S}_1| = |\mathcal{T}_1|$.

Let α be the largest element in \mathcal{S}_2 . We show that there is an injective mapping from $\mathcal{S}_2 - \{\alpha\}$ to \mathcal{T}_2 . This will give $|\mathcal{S}_2| - 1 \leq |\mathcal{T}_2|$, and therefore completes the proof.

By the definition of \mathcal{S}_2 , we have $\gamma_{n_1} > \gamma_{n_2}$ if $n_1, n_2 \in \mathcal{S}_2$ and $n_1 > n_2$. Thus, if $n \in \mathcal{S}_2 - \{\alpha\}$, then $\gamma_n < \gamma_\alpha$, which implies that $\gamma_n^+ \leq \gamma_\alpha \leq 2T$.

Take any $n \in \mathcal{S}_2 - \{\alpha\}$. It follows from the definition of \mathcal{S}_2 that $\gamma_n^+ - \gamma_n \leq v/\log T \leq 2v/\log \gamma_n$. Applying Theorem 2, we see that the box

$$\{s = \sigma + it : 1/2 < \sigma < 1/2 + v^2/\log \gamma_n, \gamma_n < t < \gamma_n^+\}$$

contains a zero of $\zeta'(s)$, say $\rho'_{k(n)}$. Since

$$0 < \beta'_{k(n)} - \frac{1}{2} < \frac{v^2}{\log \gamma_n} \leq \frac{v^2}{\log T}, \quad \text{and} \quad T \leq \gamma_n < \gamma'_{k(n)} < \gamma_n^+ \leq 2T,$$

it follows that $k(n) \in \mathcal{T}_2$.

We take the mapping $\phi : \mathcal{S}_2 - \{\alpha\} \rightarrow \mathcal{T}_2$ via $\phi(n) = k(n)$. It remains to show that ϕ is injective. Suppose this is not the case, then we would have $k(n_1) = k(n_2)$ for some $n_1 > n_2$. But that would imply

$$\gamma'_{k(n_1)} > \gamma_{n_1} \geq \gamma_{n_2}^+ > \gamma'_{k(n_2)} = \gamma'_{k(n_1)},$$

a contradiction. This completes the proof. □

Proof of Theorem 3. Let $\rho' = \beta' + i\gamma'$ be a zero of $\zeta'(s)$ with $\beta' > 1/2$ and γ' large. Let γ and γ^+ be such that $\gamma < \gamma' < \gamma^+$.

By Lemma 8 we have

$$\sum_{1/2 < \Re \lambda'} \theta(\lambda', \gamma, \gamma^+) + \frac{\log 2}{2}(\gamma^+ - \gamma) + O\left(\frac{\gamma^+ - \gamma}{\gamma}\right) = \pi,$$

where the sum is over all zeros λ' of $\zeta'(s)$ with real part greater than $1/2$. In particular, this gives

$$\theta(\rho', \gamma, \gamma^+) < \pi - \frac{\log 2}{2}(\gamma^+ - \gamma) + O\left(\frac{\gamma^+ - \gamma}{\gamma}\right).$$

Thus, it follows that

$$\theta_1 + \theta_2 > \frac{\log 2}{2}(\gamma^+ - \gamma) + O\left(\frac{\gamma^+ - \gamma}{\gamma}\right) \gg \gamma^+ - \gamma,$$

where θ_1 and θ_2 are the angles of at ρ and ρ^+ , respectively, of the triangle (ρ, ρ', ρ^+) .

By Lemma 9

$$|\rho - \rho'| \gg \sqrt{(\beta' - 1/2)/\log \gamma}.$$

Therefore, we see that

$$\sin \theta_1 = \frac{\beta' - 1/2}{|\rho - \rho'|} \ll \sqrt{(\beta' - 1/2) \log \gamma},$$

and this gives

$$\theta_1 \ll \sqrt{(\beta' - 1/2) \log \gamma}.$$

Similarly, we have

$$\theta_2 \ll \sqrt{(\beta' - 1/2) \log \gamma}.$$

Thus, we see that

$$\gamma^+ - \gamma \ll \theta_1 + \theta_2 \ll \sqrt{(\beta' - 1/2) \log \gamma}.$$

This proves the theorem. □

4. PROOFS OF LEMMAS 4, 5, 6, 7 AND 8

Proof of Lemma 4. Following Zhang [19], we let $\xi(s) = h(s)\zeta(s)$ and $\eta(s) = h(s)\zeta'(s)$, where $h(s) = \pi^{-s/2}\Gamma(s/2)$. Note that for any integer $n > 0$, we have

$$i^n \xi^{(n)}(1/2 + it) \in \mathbb{R}$$

by the functional equation.

Suppose that $\rho = 1/2 + i\gamma$ is a zero of $\zeta(s)$ with multiplicity m . Then we have

$$\zeta(\rho) = \zeta'(\rho) = \cdots = \zeta^{(m-1)}(\rho) = 0, \quad \zeta^{(m)}(\rho) \neq 0.$$

It follows from Leibniz' law that

$$\xi(\rho) = \xi'(\rho) = \cdots = \xi^{(m-1)}(\rho) = \eta(\rho) = \eta'(\rho) = \cdots = \eta^{(m-2)}(\rho) = 0, \quad \xi^{(m)}(\rho) = \eta^{(m-1)}(\rho) \neq 0.$$

In particular, we see that $i^m \eta^{(m-1)}(\rho) \in \mathbb{R}$.

Write $\eta(x + iy) = \eta(x, y)$. If $\eta(s)$ is holomorphic at $s = x + iy$, then we have

$$\begin{aligned}\eta'(x + iy) &= \eta_y(x, y) \cdot (i)^{-1}, \\ \eta^{(2)}(x + iy) &= \eta_{yy}(x, y) \cdot (i)^{-2}, \\ &\dots \\ \eta^{(k)}(x + iy) &= \eta_{y^k}(x, y) \cdot (i)^{-k}, \quad \forall k \in \mathbb{Z}^+.\end{aligned}\tag{10}$$

Now since $\eta(\rho) = \eta'(\rho) = \dots = \eta^{(m-2)}(\rho) = 0$ and $\eta^{(m-1)}(\rho) \neq 0$, it follows that

$$\eta(1/2, \gamma) = \eta_y(1/2, \gamma) = \dots = \eta_{y^{m-2}}(1/2, \gamma) = 0, \quad \eta_{y^{m-1}}(1/2, \gamma) \neq 0.$$

Therefore, when t is in a neighborhood of γ in which $\eta(s)$ is holomorphic, we can write

$$\eta(1/2, t) = (t - \gamma)^{m-1} \cdot p(t)\tag{11}$$

for some function p with $p(\gamma) \neq 0$. Hence, we have

$$\begin{aligned}\eta'(1/2 + it) &= \eta_y(1/2, t) \cdot i^{-1} \\ &= \left((t - \gamma)^{m-1} p'(t) + (m-1)(t - \gamma)^{m-2} p(t) \right) \cdot i^{-1}.\end{aligned}$$

This gives

$$\frac{\eta'}{\eta}(1/2 + it) = i^{-1} \frac{p'}{p}(t) + i^{-1}(m-1)(t - \gamma)^{-1},$$

and in particular,

$$-\Re \frac{\eta'}{\eta}(1/2 + it) = -\Im \frac{p'}{p}(t).$$

From this and the definition of $F(t)$, we see that

$$F(\gamma) = -\Im \frac{p'}{p}(\gamma)$$

since $p(\gamma) \neq 0$.

Now Leibniz' law gives us

$$\xi^{(m+1)}(\rho) = h\zeta^{(m+1)}(\rho) + (m+1)h'\zeta^{(m)}(\rho)$$

and

$$\xi^{(m)}(\rho) = h\zeta^{(m)}(\rho).$$

It follows that

$$\frac{\xi^{(m+1)}}{\xi^{(m)}(\rho)} = \frac{\zeta^{(m+1)}}{\zeta^{(m)}(\rho)} + (m+1) \frac{h'}{h}(\rho).$$

Similarly, we obtain

$$\frac{\eta^{(m)}}{\eta^{(m-1)}(\rho)} = \frac{\zeta^{(m+1)}}{\zeta^{(m)}(\rho)} + m \frac{h'}{h}(\rho).$$

This gives

$$\frac{\xi^{(m+1)}}{\xi^{(m)}(\rho)} - \frac{\eta^{(m)}}{\eta^{(m-1)}(\rho)} = \frac{h'}{h}(\rho).$$

On the other hand, by (10) and (11) we can easily compute that

$$\frac{\eta^{(m)}}{\eta^{(m-1)}(\rho)} = i^{-1} m \frac{p'}{p}(\gamma).$$

Combining the above two formulas, and taking the real part, we see that

$$F(\gamma) = \frac{1}{m} \Re \frac{h'}{h}(\rho) = \frac{1}{2m} \log \gamma + O(1/m),$$

where we apply Stirling's formula in the last equality. This completes the proof of Lemma 4. \square

Proof of Lemma 5. The case that $\rho = 1/2 + i\gamma$ is a simple zero of $\zeta(s)$ has been treated in [19]. Below we assume that $\rho = 1/2 + i\gamma$ is a multiple zero of $\zeta(s)$. Namely, we assume $\eta(1/2 + i\gamma) = 0$.

We use a temporary notation \lim to denote $\lim_{\substack{v \rightarrow \gamma \\ v > \gamma}}$ (or $\lim_{\substack{v \rightarrow \gamma \\ v < \gamma}}$). It is easy to see that

$$\begin{aligned} \lim \arg \eta(1/2 + it_1) &\equiv \lim \arctan \frac{\Im \eta(1/2 + it_1)}{\Re \eta(1/2 + it_1)} \pmod{\pi} \\ &\equiv \arctan \lim \frac{\Im \eta(1/2 + it_1)}{\Re \eta(1/2 + it_1)} \pmod{\pi}. \end{aligned}$$

By (11) we have

$$\Re \eta(1/2, t_1) = (t_1 - \gamma)^{m-1} \cdot \Re p(t_1)$$

and

$$\Im \eta(1/2, t_1) = (t_1 - \gamma)^{m-1} \cdot \Im p(t_1).$$

It follows that

$$\begin{aligned} \lim \arg \eta(1/2 + it_1) &\equiv \arctan \lim \frac{\Im \eta(1/2 + it_1)}{\Re \eta(1/2 + it_1)} \pmod{\pi} \\ &\equiv \arctan \lim \frac{(t_1 - \gamma)^{m-1} \cdot \Im p(t_1)}{(t_1 - \gamma)^{m-1} \cdot \Re p(t_1)} \pmod{\pi} \\ &\equiv \arctan \lim \frac{\Im p(t_1)}{\Re p(t_1)} \pmod{\pi}. \end{aligned}$$

Recall that $i^m \eta^{(m-1)}(\rho) \in \mathbb{R}$ and that $\eta_{y^{m-1}}(1/2, \gamma) \cdot (i)^{-(m-1)} = \eta^{(m-1)}(\rho)$. Thus, we have

$$i \cdot \eta_{y^{m-1}}(1/2, \gamma) \in \mathbb{R}.$$

Since $\eta_{y^{m-1}}(1/2, \gamma) = (m-1)! p(\gamma)$, we have $ip(\gamma) \in \mathbb{R}$, namely, $\Re p(\gamma) = 0$. From this and the fact that $p(\gamma) \neq 0$, we see that $\Im p(\gamma) \neq 0$. Therefore, we have

$$\begin{aligned} \lim \arg \eta(1/2 + it_1) &\equiv \arctan \lim \frac{\Im p(t_1)}{\Re p(t_1)} \pmod{\pi} \\ &\equiv \pi/2 \pmod{\pi}. \end{aligned}$$

\square

Proof of Lemma 6. By Zhang's result (i), we have

$$\int_{\gamma}^{\gamma^+} F(t)dt = \lim \int_{t_1}^{t_2} F(t)dt, \quad (12)$$

where the limit is taken for $t_1 > \gamma, t_1 \rightarrow \gamma$ and $t_2 < \gamma^+, t_2 \rightarrow \gamma^+$. By equation (2.16) in [19]

$$\int_{t_1}^{t_2} F(t)dt = \arg \eta(1/2 + it_1) - \arg \eta(1/2 + it_2). \quad (13)$$

It follows that

$$\int_{\gamma}^{\gamma^+} F(t)dt = \lim \arg \eta(1/2 + it_1) - \lim \arg \eta(1/2 + it_2).$$

By Lemma 5, this is

$$\int_{\gamma}^{\gamma^+} F(t)dt \equiv 0 \pmod{\pi}. \quad (14)$$

By Zhang's result (ii),

$$F(t) = F_1(t) - F_2(t) + O(1).$$

This comes from considering the Hadamard factorization for $\eta(s)$. With a little more care we may actually get

$$F(t) = F_1(t) - F_2(t) - C + O(1/t),$$

for some constant C , and on RH this is

$$F(t) = F_1(t) - C + O(1/t). \quad (15)$$

The expression for C is given by

$$C = \Re \sum_{\beta' > 0} \frac{1}{\rho'} + \sum_{n=1}^{\infty} \left(\frac{1}{\rho'_n} + \frac{1}{2n} \right) - \frac{\log \pi}{2} - \frac{C_0}{2} - 2 + \frac{\zeta''}{\zeta'}(0), \quad (16)$$

where C_0 is Euler's constant, and $\rho'_n \in (-2n - 2, -2n)$ is a real zero of $\zeta'(s)$ (see Theorem 9 in [12]).

One can easily show that $C < 0$ (unconditionally). In fact, by equation (4) in [18], we have

$$\Re \sum_{\beta' > 0} \frac{1}{\rho'} < 0.185.$$

Also, it is easy to calculate that

$$\sum_{n=1}^{\infty} \left(\frac{1}{2n} + \frac{1}{\rho'_n} \right) < 0.32.$$

Inserting the values for other constants in (16), we get $C < -0.17$.

By (15) we see that $F(t) > F_1(t) \geq 0$ for large t . It follows that

$$\int_{\gamma}^{\gamma^+} F(t)dt > 0.$$

Combining this with (8) and (14), we obtain

$$\int_{\gamma}^{\gamma^+} F(t)dt = \pi$$

for all large γ . □

Proof of Lemma 7. Consider the integral

$$\int_T^{2T} F(t)dt.$$

Ordering the ordinates $T \leq g_1 < g_2 < \cdots < g_{N_d} \leq 2T$ of distinct zeros of $\zeta(s)$ in $[T, 2T]$, we can write the above integral as

$$\int_T^{g_1} + \int_{g_1}^{g_2} + \cdots + \int_{g_{N_d-1}}^{g_{N_d}} + \int_{g_{N_d}}^{2T} F(t)dt = \sum_{j=1}^{N_d} \int_{g_j}^{g_{j+1}} F(t)dt + O(1). \quad (17)$$

On the other hand, by (15) we have

$$\int_T^{2T} F(t)dt = \int_T^{2T} F_1(t)dt - CT + O(1).$$

By the definition of $F_1(t)$,

$$\int_T^{2T} F_1(t)dt = -\Re \sum_{\beta' > 1/2} \int_T^{2T} \frac{1}{1/2 + it - \rho'} dt = \sum_{\beta' > 1/2} \theta(\rho', T, 2T).$$

Hence, we obtain

$$\int_T^{2T} F(t)dt = \sum_{1/2 < \beta'} \theta(\rho', T, 2T) - CT + O(1).$$

By standard estimates, one easily shows that

$$\sum_{1/2 < \beta'} \theta(\rho', T, 2T) = \pi N_d - T \log 2/2 + O(T/\log T).$$

It follows that

$$\int_T^{2T} F(t)dt = \pi N_d - T \log 2/2 - CT + O(T/\log T).$$

This together with (17) gives

$$\sum_{j=1}^{N_d} \left(\pi - \int_{g_j}^{g_{j+1}} F(t)dt \right) = \left(C + \frac{\log 2}{2} \right) T + O(T/\log T).$$

By Lemma 6, the left-hand side is 0 for all large T . Thus we have $C = -\log 2/2$. The result now follows from (15). □

Proof of Lemma 8. The result follows immediately from Lemma 6 and Lemma 7. □

5. PROOF OF LEMMA 11

Proof of Lemma 11. Let γ be large. Let $\mathcal{O} = 1/2 + i(\gamma + \gamma^+)/2$. That is, \mathcal{O} is the middle point of ρ and ρ^+ . Write $\gamma^+ - \gamma = 2l$ and $d = 5l$. Also write

$$F_1(t) = F_{11}(t) + F_{12}(t), \quad (18)$$

where

$$F_{11}(t) = - \sum_{\substack{\beta' > 1/2 \\ |\rho' - \mathcal{O}| < d}} \Re \frac{1}{1/2 + it - \rho'} = \sum_{\substack{\beta' > 1/2 \\ |\rho' - \mathcal{O}| < d}} \frac{\beta' - 1/2}{(\beta' - 1/2)^2 + (\gamma' - t)^2},$$

and

$$F_{12}(t) = - \sum_{\substack{\beta' > 1/2 \\ |\rho' - \mathcal{O}| \geq d}} \Re \frac{1}{1/2 + it - \rho'} = \sum_{\substack{\beta' > 1/2 \\ |\rho' - \mathcal{O}| \geq d}} \frac{\beta' - 1/2}{(\beta' - 1/2)^2 + (\gamma' - t)^2}.$$

For $|\rho' - \mathcal{O}| \geq d$ and $\gamma \leq t \leq \gamma^+$, we have

$$|\rho' - (1/2 + it)| \geq |\rho' - \rho| - |\rho - (1/2 + it)| \geq |\rho' - \rho|/2$$

and

$$|\rho' - (1/2 + it)| \leq |\rho' - \rho| + |\rho - (1/2 + it)| \leq 3|\rho' - \rho|/2.$$

It follows that

$$4F_{12}(\gamma)/9 \leq F_{12}(t) \leq 4F_{12}(\gamma).$$

In particular, this gives

$$F_{12}(t) \leq 2 \log \gamma + O(1) \quad (19)$$

in view of Lemma 4.

Now suppose that there is no zero of $\zeta'(s)$ in the box

$$\{s = \sigma + it : 1/2 < \sigma < 1/2 + 2.5(\gamma^+ - \gamma), \gamma < t < \gamma^+\}. \quad (20)$$

Then we may write

$$F_{11}(t) = f(t) + g(t),$$

where

$$f(t) = \sum_{\substack{\beta' > 1/2 \\ |\rho' - \mathcal{O}| < d \\ \gamma' \geq \gamma^+}} \frac{\beta' - 1/2}{(\beta' - 1/2)^2 + (\gamma' - t)^2},$$

and

$$g(t) = \sum_{\substack{\beta' > 1/2 \\ |\rho' - \mathcal{O}| < d \\ \gamma' \leq \gamma}} \frac{\beta' - 1/2}{(\beta' - 1/2)^2 + (\gamma' - t)^2}.$$

Observe that for $\gamma \leq t \leq \gamma^+$ we have $f(t) \leq f(\gamma^+)$ and $g(t) \leq g(\gamma)$. It follows that

$$F_{11}(t) \leq f(\gamma^+) + g(\gamma) \leq F_1(\gamma^+) + F_1(\gamma) \leq \log \gamma + O(1).$$

Applying Lemma 4, we obtain

$$F_{11}(t) \leq \log \gamma + O(1). \quad (21)$$

Combining (18), (21) and (19), we get

$$F_1(t) \leq 3 \log \gamma + O(1)$$

for $t \in [\gamma, \gamma^+]$. It follows that

$$\int_{\gamma}^{\gamma^+} F_1(t) dt \leq 3(\gamma^+ - \gamma) \log \gamma + O(\gamma^+ - \gamma).$$

On the other hand, by Lemma 6 we have

$$\int_{\gamma}^{\gamma^+} F(t) dt = \pi,$$

and by Lemma 7 this is

$$\int_{\gamma}^{\gamma^+} F_1(t) dt = \pi + O(\gamma^+ - \gamma).$$

Hence, we obtain

$$\pi \leq 3(\gamma^+ - \gamma) \log \gamma + O(\gamma^+ - \gamma).$$

But since we are assuming $\gamma^+ - \gamma < a / \log \gamma$ (with $a < \pi/3$ a constant), the above inequality would give

$$\pi \leq (3 + O(\log^{-1} \gamma))a.$$

This can not hold if γ is large enough (depending on a). Hence the assumption (20) is false. This completes our proof. \square

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